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NUMERICAL STUDIES OF SLOW VISCOUS ROTATING FLOW PAST A SPHERE. III

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SUMMARY

The flow of steady incompressible viscous fluid rotating about the z-axis with angular velocity ω and moving with velocity u past a sphere of radius a which is kept fixed at the origin is investigated by means of a numerical method for small values of the Reynolds number Re_ω. The Navier–Stokes equations governing the axisymmetric flow can be written as three coupled non-linear partial differential equations for the streamfunction, vorticity and rotational velocity component. Central differences are applied to the partial differential equations for solution by the Peaceman–Rachford ADI method, and the resulting algebraic equations are solved by the 'method of sweeps'.

The results obtained by solving the non-linear partial differential equations are compared with the results obtained by linearizing the equations for very small values of Re_ω. Streamlines are plotted for $\psi = 0{\cdot}05, 0{\cdot}2, 0{\cdot}5$ for both linear and non-linear cases. The magnitude of the vorticity vector near the body, i.e. at $z = 0{\cdot}2$, is plotted for $Re_\omega = 0{\cdot}05$, $0{\cdot}24$, $0{\cdot}5$. The correction to the Stokes drag as a result of rotation of the fluid is calculated.

KEY WORDS Peaceman–Rachford ADI method Method of sweeps Central differences of $o(h^2; k^2)$ Rotating viscous fluid

1. INTRODUCTION

Several different two-dimensional finite difference schemes can be used to approximate the Navier–Stokes equations, depending on the boundary conditions, and these schemes vary considerably in accuracy and efficiency. Central differences can be used to approximate non-linear terms, but difficulty may then be encountered in solving the finite difference equations by iterative techniques, which may fail to converge unless under-relaxation is used, particularly for high Reynolds numbers. Upwind and downwind differences are used in approximating non-linear terms. Such approximation improves the convergence of iterative procedures of solution, but they are only first-order accurate. The convergence is improved because the matrices associated with the finite difference equations are diagonally dominant. Numerical calculations were performed by Dennies *et al.*[1] using a two-dimensional specialized finite difference scheme and series truncation method. The calculations were carried out with grid sizes of $\pi/30$ and $\pi/60$ in the angular direction, and in the radial direction a grid size as small as $0{\cdot}01$ was used. The Gauss–Seidel iterative method was used in all the iterative procedures without under-relaxation. The starting values were taken as the results for $T=0$ computed by Dennis and Ingham.[2]

In this paper, the flow of steady incompressible viscous fluid rotating about the z-axis with angular velocity ω and moving with velocity u along the z-axis past a sphere is considered. For

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small values of Re_ω, the linear and non-linear problems are equivalent. Both these cases are solved by the Peaceman–Rachford ADI method. The starting values for ψ and Ω are taken as zero. The streamlines are drawn for $Re_\omega=0{\cdot}05$, 0·24, 0·5 and the effects of rotation on the Stokes drag for both linear and non-linear cases are compared. The magnitudes of the vorticity vector at $z=0{\cdot}2$ for both cases are drawn and compared.

2. FORMULATION OF THE PROBLEM

We consider a steady slow viscous fluid which is in solid body rotation with angular velocity ω about the z-axis and moves with uniform velocity u along the z-direction. A sphere of radius a is introduced in the flow and kept fixed at the origin. We use spherical polar co-ordinates (r, θ, ϕ); since the motion is axially symmetric, all quantities are independent of ϕ. If the transformation $r=e^z$ is used, the Navier–Stokes equations can be expressed in the form

$$D^2\psi=-e^{3z}\zeta\sin\theta=-e^{2z}\zeta_1, \tag{1}$$

$$D^2\zeta=\frac{Re_\omega e^{-z}}{\sin\theta}\left[\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta_1}{\partial z}-\frac{\partial\psi}{\partial z}\frac{\partial\zeta_1}{\partial\theta}\right)\right]+2\left(\cot\theta\frac{\partial\psi}{\partial z}-\frac{\partial\psi}{\partial\theta}\right)\zeta_1-2\Omega\left(\cot\theta\frac{\partial\Omega}{\partial z}-\frac{\partial\Omega}{\partial\theta}\right), \tag{2}$$

$$D^2\Omega=\frac{Re_\omega e^{-z}}{\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\Omega}{\partial z}-\frac{\partial\psi}{\partial z}\frac{\partial\Omega}{\partial\theta}\right), \tag{3}$$

where

$$D^2\equiv\frac{\partial^2}{\partial z^2}-\frac{\partial}{\partial z}+\frac{\sin\theta}{z^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

Here ψ, Ω and ζ_1 are the dimensionless streamfunction, the rotational velocity component and the vorticity respectively. The non-dimensional variables are

$$r=r'/a, \qquad Re=Ga^3/\nu, \qquad C=2a\omega/\nu.$$

The velocity components are given by

$$v_r=\frac{e^{-2z}}{\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad v_\theta=-\frac{e^{-2z}}{\sin\theta}\frac{\partial\psi}{\partial z}.$$

Equations (1)–(3) are to be solved with the following boundary conditions:

$$\Omega=\psi=\partial\psi/\partial z=0 \quad \text{on } z=0, \tag{4}$$

$$\left.\begin{aligned}\psi&\sim(1/2)e^{2z}\sin^2\theta\\ \Omega&\sim(C/2)e^{2z}\sin^2\theta\end{aligned}\right\}\quad\text{for sufficiently large distances,} \tag{5}$$

$$\psi=0 \quad \text{for } \theta=0,\ 180^\circ\text{—axis of symmetry,} \tag{6}$$

$$\zeta=0 \quad \text{for } \theta=0,\ 180^\circ\text{—axis of symmetry,} \tag{7}$$

$$\zeta\to 0 \quad \text{at sufficiently large distances.} \tag{8}$$

The condition for ζ at the surface of the sphere comes out of $\partial\psi/\partial z=0$.

3. FINITE DIFFERENCE EQUATIONS

Central differences of order h^2 and k^2 are written for equations (1)–(3) and the boundary conditions (4)–(8). This is explained in detail in Reference 3 and is not repeated here. The finite difference boundary conditions are

$$\Omega_{0,j}=\psi_{0,j}=(\partial\psi/\partial z)_{0,j}=0 \quad \text{on } z=0 \quad \text{(sphere)} \tag{9}$$

A sufficiently large distance is taken as e^2, i.e. $z_N=2$. Thus

$$\left.\begin{aligned}&\psi_{N,j}\sim(1/2)e^4\sin^2\theta_j\\&\Omega_{N,j}\sim(C/2)e^4\sin^2\theta_j\\&\zeta_{1N,j}\sim 0\end{aligned}\right\}\quad \text{at } z=Z_N=2, \tag{10}$$

$$\zeta_{0,j}=-\frac{8\psi_{1,j}-\psi_{2,j}}{2h^2\sin\theta_j}\quad \text{on } z=0. \tag{11}$$

The three coupled equations (1)–(3) with the boundary conditions (9)–(11) are solved using the Peaceman–Rachford ADI method. In this method, to ensure diagonal dominance we choose the acceleration parameter ρ as follows:

$$\begin{aligned}&\rho=30 && \text{for iterations 1–5,}\\&\rho=30 && \text{for iterations 6–11,}\\&\rho=110 && \text{for iterations 12–21,}\\&\rho=130 && \text{for iterations 22–31.}\end{aligned}$$

The numbers of iterations required to ensure convergence are given in Table I.

The procedure adopted is as follows.

(i) The starting values of ψ, ζ_1 and Ω are taken as zero.
(ii) The equation for Ω is iterated 15 times. The final value of Ω is taken as the initial value for further computation.
(iii) The final Ω-value computed in step (ii) is used in the equation for ζ_1, which is iterated 15 times. The final value of ζ_1 is taken as the initial value for further computation.
(iv) The final ζ_1-value computed in step (iii) is used in the equation for ψ. The value of ψ computed serves as the initial value for further computation.

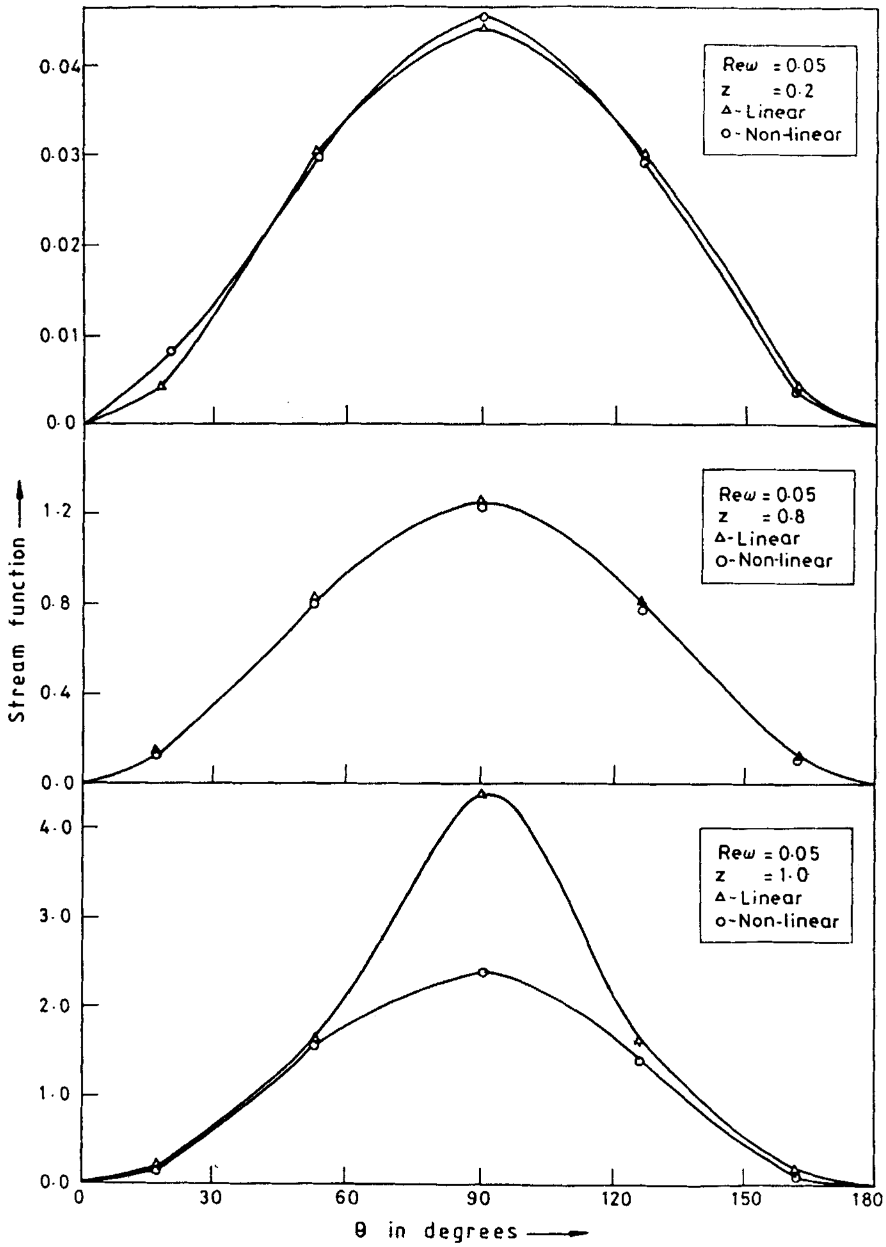
This process had to be repeated six times for two successive iterated Ω-values to be convergent; for this, steps (ii) and (iii) had to be repeated 16 times to ensure convergence.

There is a good coincidence between horizontal and vertical sweeps at the point (z_i, θ_j), $i, j=1, 2, \ldots, 9$. A computer program for the Peaceman–Rachford ADI method has been developed on the IBM 370/155. The resulting algebraic equations are solved by the 'method of sweeps' by dividing them into nine blocks, each of which is a tridiagonal system.

The method of solving is introduced as four subroutines in the computer programs developed. The first three subroutines are for ψ, ζ_1 and Ω; the fourth subroutine is introduced for the 'method of sweeps'.

Table I

Equation	Number of iterations
(1) for ψ	0
(3) for Ω	15
(2) for ζ_1	15



Reω = 0·05
z = 0·2
Δ - Linear
o - Non-linear
Reω = 0·05
z = 0·8
Δ - Linear
o - Non-linear
Reω = 0·05
z = 1·0
Δ - Linear
o - Non-linear
Stream function
θ in degrees
0·0
0·01
0·02
0·03
0·04
0·4
0·8
1·2
1·0
2·0
3·0
4·0
0
30
60
90
120
150
180

Figure 1

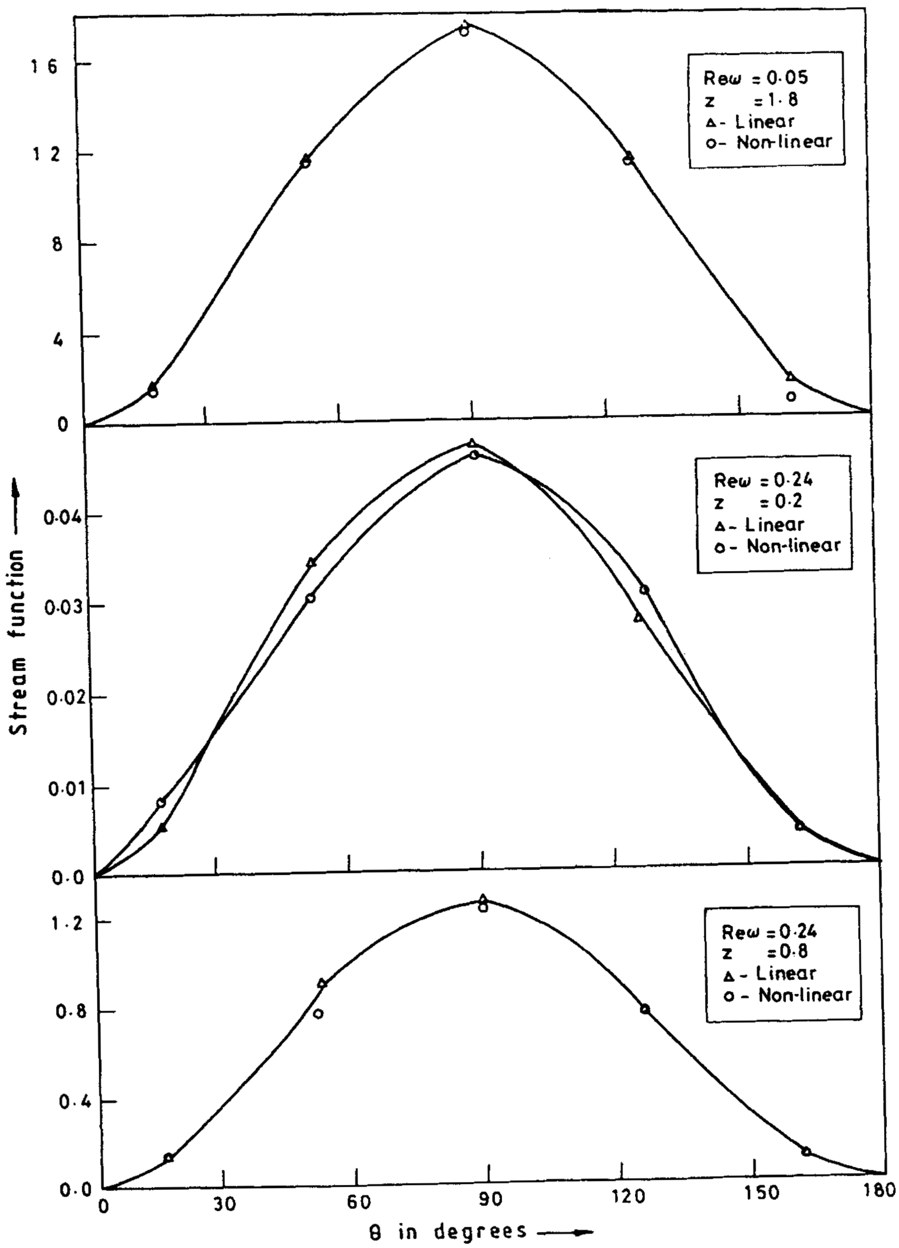
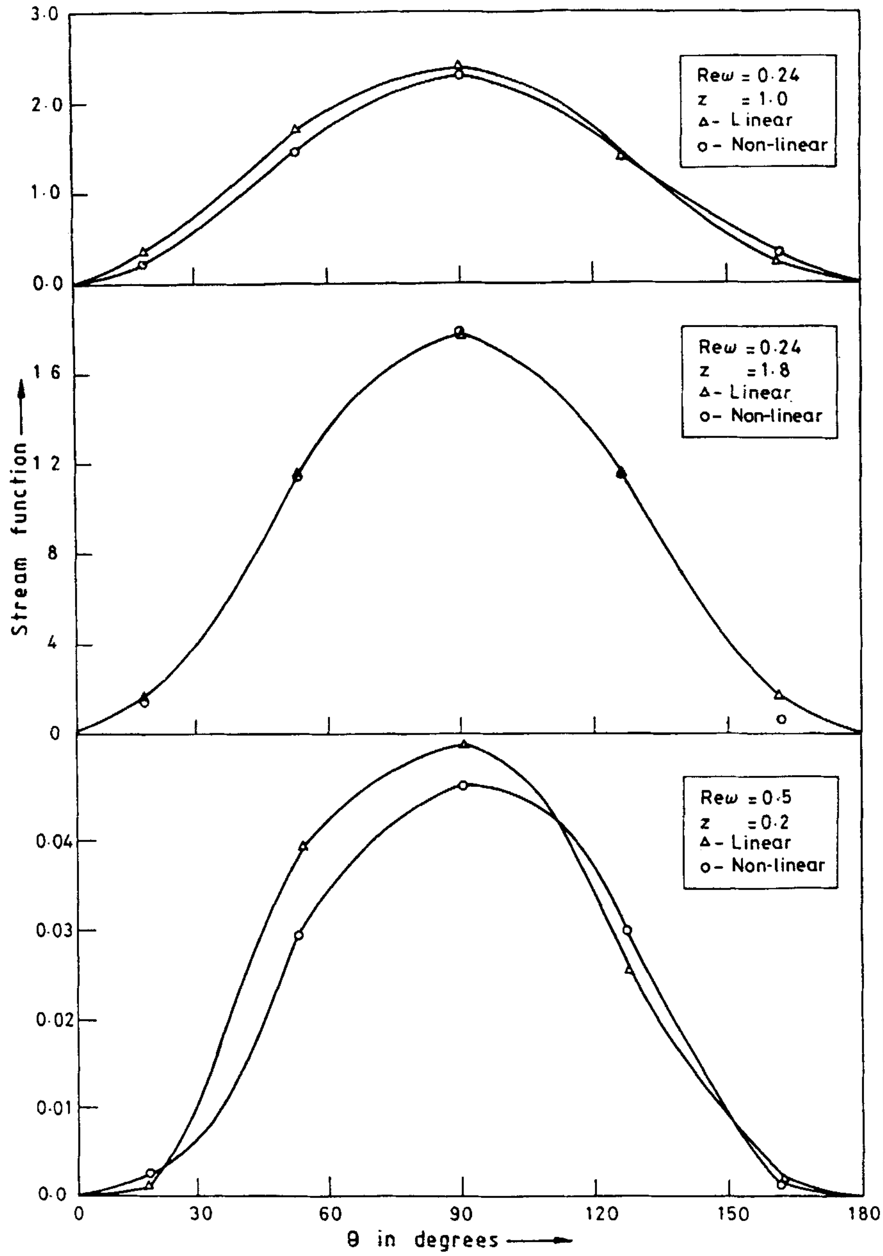

Reω = 0·05
z = 1·8
Δ- Linear
o- Non-linear
Reω = 0·24
z = 0·2
Δ- Linear
o- Non-linear
Reω = 0·24
z = 0·8
Δ - Linear
o - Non-linear
Stream function
θ in degrees
16
12
8
4
0
0·04
0·03
0·02
0·01
0·0
1·2
0·8
0·4
0·0
0
30
60
90
120
150
180

Figure 2



Stream function
θ in degrees
Reω = 0·24
z = 1·0
Δ- Linear
o - Non-linear
Reω = 0·24
z = 1·8
Δ- Linear
o- Non-linear
Reω = 0·5
z = 0·2
Δ- Linear
o- Non-linear
3·0
2·0
1·0
0·0
1 6
1 2
8
4
0
0·04
0·03
0·02
0·01
0·0
0
30
60
90
120
150
180

Figure 3

4. DISCUSSION OF RESULTS

For small Reynolds numbers, the linear and non-linear problems represent the same flow and hence comparison is made between the two cases. For $Re_\omega = 0{\cdot}05$, $0{\cdot}24$ and $0{\cdot}5$ the effects of rotation on the Stokes drag D_S for the linear and non-linear problems are given in Table II.

The drag on the sphere is given by the formula

$$D = \frac{D_S}{3} \int_0^\pi \left[\left(-p + 2e^{-z} \frac{\partial v_r}{\partial z} \right) \cos\theta - \left(e^{-z} \frac{\partial v_\theta}{\partial z} - v_\theta + \frac{\partial v_r}{\partial \theta} \right) \sin\theta \right] e^{2z} \sin\theta \, d\theta \quad \text{at } z = 0,$$

Table II

Re_ω	Linear drag	Non-linear drag
0·05	1·163709 D_S	1·234675 D_S
0·24	1·211938 D_S	1·234675 D_S
0·5	1·326986 D_S	1·23755 D_S

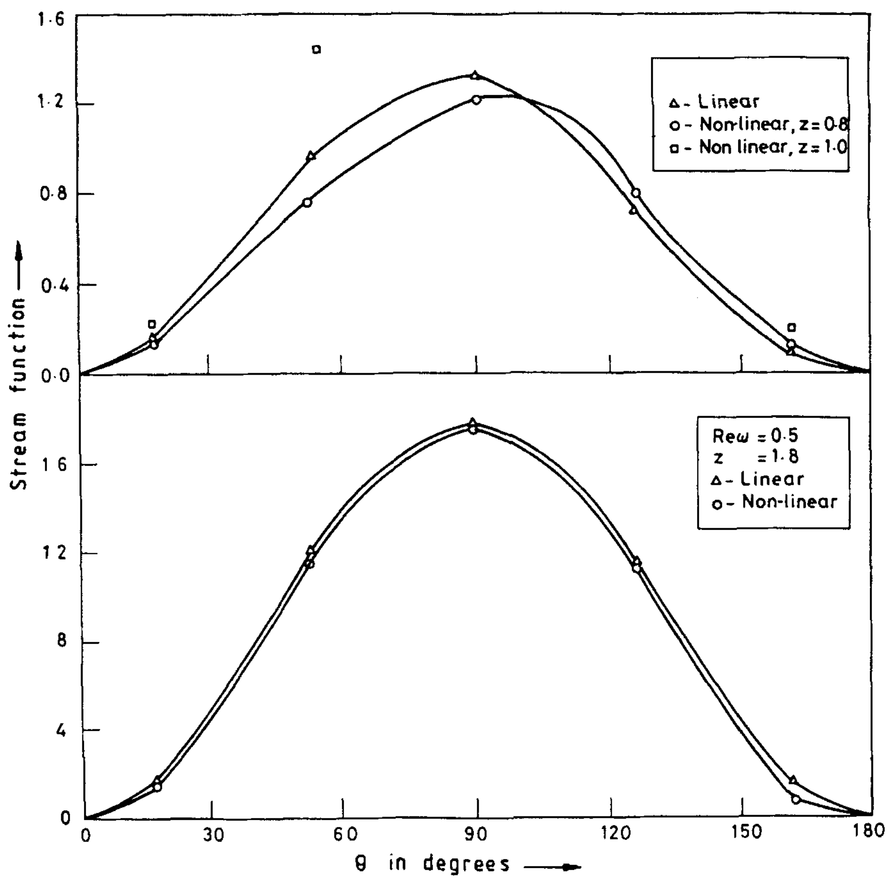


Figure 4

where

$$p=\int_0^\theta \left(e^{-2z}\frac{\partial^2(e^z v_\theta)}{\partial z^2}\right)d\theta \quad \text{at } z=0.$$

D_S is the drag for Stokes flow, $D_S=6\pi\mu a u$, and p is the dimensionless pressure in the fluid. It is seen that the effect of rotation is to increase the drag as Re_ω increases.

The variations of the streamfunction in the linear and non-linear cases with increasing θ for the values $z=0{\cdot}2$, $0{\cdot}8$, $1{\cdot}0$, $1{\cdot}8$ and $Re_\omega=0{\cdot}05$, $0{\cdot}24$, $0{\cdot}5$ are plotted in Figures 1–4.

The magnitude of the vorticity vector for $z=0{\cdot}2$ with increasing θ is plotted in Figure 5 in the linear and non-linear cases. The magnitude of the vorticity vector is $\sqrt{(\xi^2+\eta^2+\zeta^2)}$, where

$$\xi=\frac{e^{-2z}}{\sin\theta}\frac{\partial\Omega}{\partial\theta}, \qquad \eta=\frac{e^{-2z}}{\sin\theta}\frac{\partial\Omega}{\partial z}, \qquad \zeta=\frac{-e^{-3z}}{\sin\theta}D^2\psi.$$

The streamlines $\psi=0{\cdot}05$, $0{\cdot}2$, $0{\cdot}5$ for $Re_\omega=0{\cdot}05$, $0{\cdot}24$, $0{\cdot}5$ in the linear and non-linear cases are plotted in Figure 6.

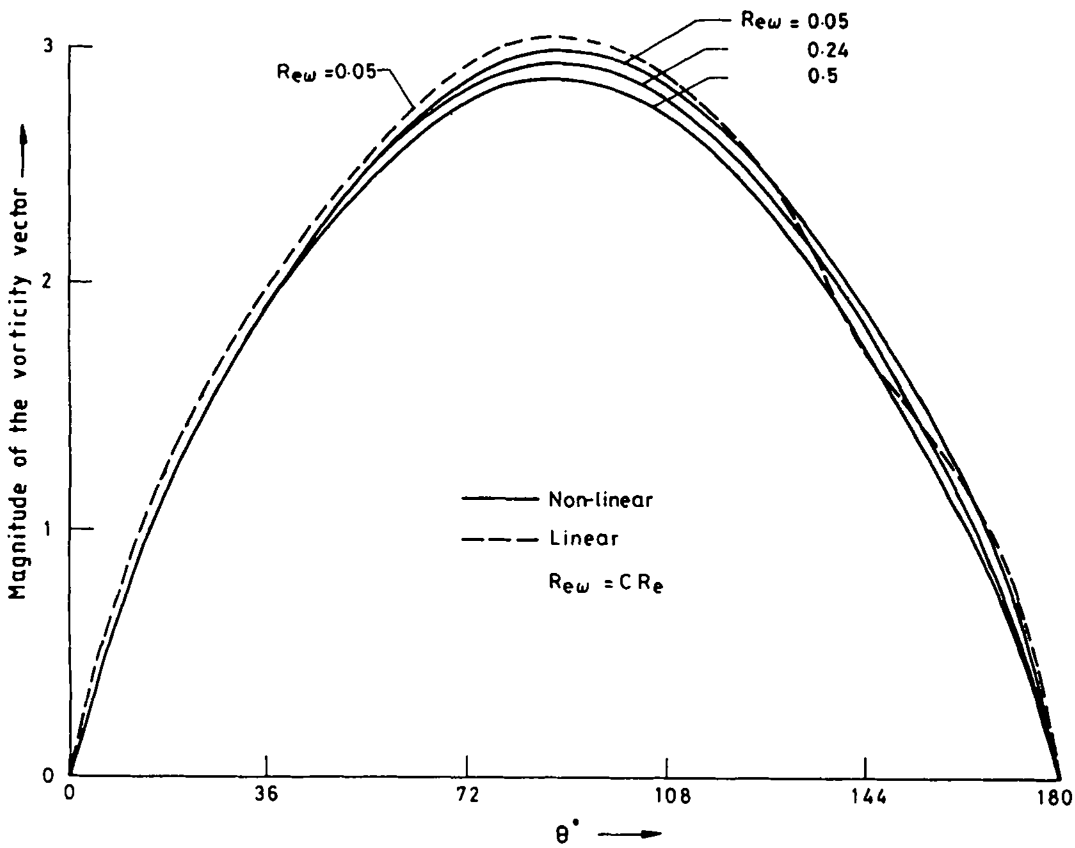


Figure 5. Magnitude of the vorticity vector for $z=0{\cdot}2$

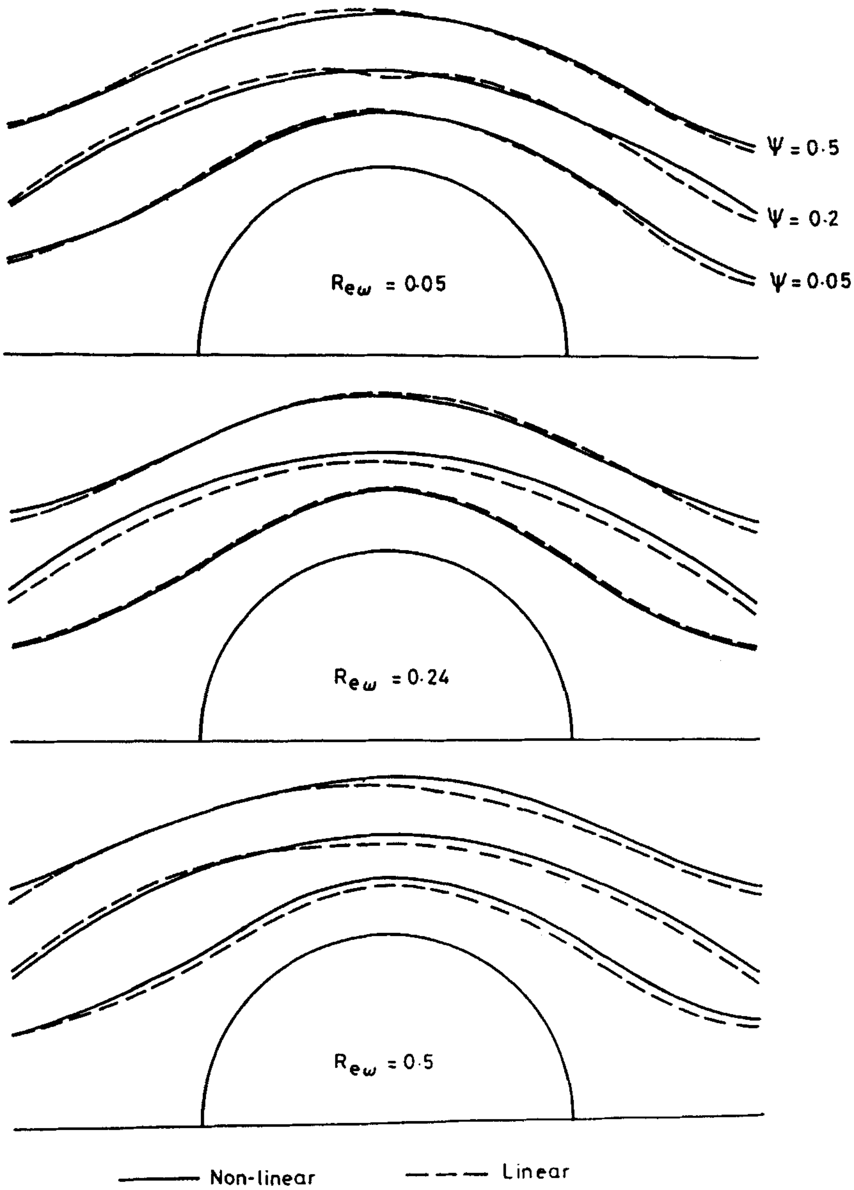


Fig. 6. Streamlines for 'method of sweeps'

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